LA 2 Final Cram

A stir-fried tomato egg is an element in the vector space of Chinese dishes, where it is made up by 3 times a chicken egg, plus 2 times a tomato, plus 1-2 times a cup of rice, plus some times a scoop of salt, and some scoop of soy sauce and sugar.

- How to Cook from 0 for Mathematicians

Guide Content:

1. Some tips and tricks
2. Typical problems in finals

As usual, if you see any errors contact me.

1 Tips and Tricks

1. **Invertibility & Diagonalizability**

   When I first started learning about diagonalizability of matrices, I was often confused about the relationship between this and invertibility. Fortunately, it turns out that those two doesn't matter. I will give the simplest example of each and memorizing such examples can be helpful for some of the past final true/false and give-example questions.

	Invertible	**Not Invertible**
Diagonalizable	$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Not Diagonalizable	$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

2. **Invertible Condition**

   Given $F : V \to V$, $\dim(V) = N$, B is any basis of V, and $N \times N$ matrix $A = [F]_B$, For the following, if A is invertible, then F is invertible.

   (a) If all N column vectors of A linearly independent, then A invertible.

   (b) If $\text{rk}(A) = N$, then A invertible.
   Since $\text{rank}(A)$ equals number of linearly independent column vectors.

   (c) If $\dim(\ker(A)) = 0$, then A invertible.
   Since $N - \dim(\ker(A)) = \text{rank}(A)$.

   (d) If $\text{rref}(A)$ have no rows of zeros, then A invertible.
   Since any rows of zeros mean $\dim(\ker(A)) > 0$.

(e) If A is bijective, then A invertible.
Since bijective means $\text{rank}(A) = N$

(f) If A is surjective, or injective, then A invertible.
Since surjective $\leftrightarrow$injective $\leftrightarrow$bijective when A is $N \times N$.

(g) If $\det(A) \neq 0$, then A invertible.
Since $\text{rref}(A)$ will have a row of zero when $\det(A) = 0$.

(h) If no eigenvalue is 0, then A invertible.
Since $\det(A)$ equals product of eigenvalues.

3. **Diagonalizable Condition**

Given $F : V \to V$, $\dim(V) = N$, B is any basis of V, and $N \times N$ matrix $A = [F]_B$, For the following, if A is diagonalizable, then F is diagonalizable.

(a) If there exists invertible matrix $S \in \mathbb{R}^{N \times N}$ and diagonal matrix D such that $S^{-1}AS = D$, then A diagonalizable.

(b) If there exists eigenbasis for A, then A diagonalizable.
Since the column vectors of S in $S^{-1}AS = D$ can just be the basis vectors.

(c) If there are N distinct eigenvalues for A, then A diagonalizable.
Since each distinct eigenvalue has its eigenvector linearly independent from others, implies eigenbasis exists.

(d) If $\text{geomu}(A) = N$, then A diagonalizable.
Since $\text{geomu}(A) = N$ implies N linearly independent eigenvectors, implies eigenbasis exists.

(e) If A is symmetrical, then A diagonalizable.
Since symmetrical matrices always have eigenbasis.

Note that, although an invertible matrix A can have the column vectors switched, and still be invertible, since rank is still N. A diagonalizable matrix B has the column vectors switched, and might no longer be diagonalizable.

4. **Properties of Transpose & Inverse**:

Given $y = f(x) = Ax = \begin{pmatrix} a_1 \dots a_m \end{pmatrix} x$, where A is a $M \times N$ matrix, and $\{a_m\}_{1 \leq m \leq M}$ are the column vectors of A.

(a) $\ker(A^T A) = \ker(A)$.

(b) $(AB)^T = B^T A^T$

(c) $(AB)^{-1} = B^{-1} A^{-1}$

5. **Special Matrices**:

Below are some special matrices that can be useful to memorize when prompted to give examples of a matrix that satisfies some conditions.

(a) **Rotation Matrix**:

i. (almost always) not symmetrical
ii. (almost always) no eigenvalues
iii. (almost always) not diagonalizable
iv. invertible
v. orthogonal

Identity Matrix:

i. symmetrical
ii. N eigenvalues of 1
iii. diagonalizable

iv. invertible
v. orthogonal

Identity Matrix $\cdot -1$:

i. symmetrical
ii. N eigenvalues of 1
iii. diagonalizable
iv. invertible
v. orthogonal

Diagonal Matrix:

i. symmetrical
ii. N eigenvalues corresponding to the numbers along diagonal
iii. diagonalizable
iv. (sometimes) invertible
v. (sometimes) orthogonal

Projection Matrix:

i. symmetrical
ii. N eigenvalue of either 1 or 0
iii. diagonalizable
iv. (almost always) not invertible
v. (almost always) not orthogonal

$A + A^T$ **Matrix**:

i. symmetrical
ii. N eigenvalues
iii. diagonalizable
iv. (sometimes) invertible
v. (sometimes) orthogonal

Matrix with eigenbasis:

i. symmetrical
ii. N eigenvalues
iii. diagonalizable
iv. invertible
v. (sometimes) orthogonal

$A^2 = 0, A \neq 0$ **Matrix**:

i. (almost always) not symmetric
ii. (almost always) $\frac{N}{2}$ eigenvalues
iii. (almost always) not diagonalizable
iv. not invertible
v. not orthogonal

2 Typical problems in finals

I try to categorize the list of past final problems into groups (e.g. Is a given set a subspace). I chose one problem from each group that I feel represents their group the most, writing a solution to each chosen problem provided with my stream of thought, as well as leaving a remark on how one can generalize the solution to its entire group of problems.

Exercise 1. Final 2023.1

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}$$

1. Compute $\det(A)$
2. Find all eigenvalues of A, and their corresponding $E_\lambda(A)$.
3. Is A diagonalizable, invertible, and/or orthogonal?
4. Compute orthogonal matrix S such that $S^T AS$ is diagonal.

Particular Solution

1. The Determinant can be most easily calculated through the Laplacian expansion.

$$\begin{aligned}
\det(\begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}) &= 1 \cdot \det(\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}) - 3 \cdot \det(\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}) + 0 \cdot \det(\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}) \\
&= 1 \cdot \det(\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}) - 3 \cdot \det(\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}) \\
&= 1 \cdot (1^2 - 4^2) - 3 \cdot (3 \cdot 1 - 4 \cdot 0) \\
&= -15 - 9 \\
&= -24
\end{aligned}$$

2. The eigenvalues are derived from the characteristic polynomial of A, which is derived from the Laplacian expansion of $\det(A - \lambda I)$.
First, we calculate the characteristic polynomial, and we get

$$\begin{aligned}
\det(\begin{pmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 4 \\ 0 & 4 & 1-\lambda \end{pmatrix}) &= (1-\lambda) \cdot \det(\begin{pmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{pmatrix}) - 3 \cdot \det(\begin{pmatrix} 3 & 4 \\ 0 & 1-\lambda \end{pmatrix}) + 0 \cdot \det(\begin{pmatrix} 3 & 1-\lambda \\ 0 & 4 \end{pmatrix}) \\
&= (1-\lambda) \cdot \det(\begin{pmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{pmatrix}) - 3 \cdot \det(\begin{pmatrix} 3 & 4 \\ 0 & 1-\lambda \end{pmatrix}) \\
&= (1-\lambda) \cdot ((1-\lambda)^2 - 4^2) - 3 \cdot (3 \cdot (1-\lambda) - 4 \cdot 0) \\
&= (1-\lambda)^3 - 16(1-\lambda) - 9(1-\lambda) \\
&= (1-\lambda)((1-\lambda)^2 - 25) \\
&= (1-\lambda)(1-\lambda+5)(1-\lambda-5) \\
&= -(\lambda-1)(\lambda+6)(\lambda-4)
\end{aligned}$$

Setting this characteristic polynomial to 0, the roots of this polynomial are the possible eigenvalues.

$$-(\lambda - 1)(\lambda - 6)(\lambda + 4) = 0$$

We see $\lambda_1 = 1, \lambda_2 = 6, \lambda_3 = -4$. Now we just need to compute the corresponding eigenspace for each eigenvalue. The $E_\lambda(A)$ is equivalent to $\ker(A - \lambda I)$.

(a) For $\lambda_1 = 1$, we have

$$A - \lambda_1 I = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

To solve for $\ker(A - \lambda_1 I)$, first we need to take $\text{rref}(A - \lambda_1 I)$ and get

$$\text{rref}(A - \lambda_1 I) = \begin{pmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From this, we see that

$$\begin{aligned} E_{\lambda_1}(A) &= \ker(A - \lambda_1 I) \\ &= \text{span}(\begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix}) \end{aligned}$$

(b) For $\lambda_2 = 6$, we have

$$A - \lambda_2 I = \begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix}$$

To solve for $\ker(A - \lambda_1 I)$, first we need to take $\text{rref}(A - \lambda_2 I)$ and get

$$\text{rref}(A - \lambda_2 I) = \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

From this, we see that

$$\begin{aligned} E_{\lambda_2}(A) &= \ker(A - \lambda_2 I) \\ &= \text{span}(\begin{pmatrix} \frac{3}{4} \\ \frac{5}{4} \\ 1 \end{pmatrix}) \end{aligned}$$

(c) For $\lambda_3 = -4$, we have

$$A - \lambda_3 I = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

To solve for $\ker(A - \lambda_3 I)$, first we need to take $\text{rref}(A - \lambda_3 I)$ and get

$$\text{rref}(A - \lambda_3 I) = \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

From this, we see that

$$\begin{aligned} E_{\lambda_3}(A) &= \ker(A - \lambda_3 I) \\ &= \text{span}(\begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ 1 \end{pmatrix}) \end{aligned}$$

3. Checking the properties of invertibility and diagonalizability in the first section of this guide, we see that

   (a) The determinant is not zero, so A is invertible.

   (b) The geometric multiplicity is equal to 3, the highest possible for a 3×3 matrix, so A is diagonalizable.

(c) The column vectors are not unit vectors, meaning the magnitude is not 1, so A is not an orthogonal matrix by definition.

4. Since A is diagonalizable, and we have found the eigenvalues and eigenbasis of this matrix, we can rewrite A as

$$S^{-1}AS = D$$

$$\implies \begin{pmatrix} -\frac{4}{3} & \frac{3}{4} & \frac{3}{4} \\ 0 & \frac{5}{4} & -\frac{5}{4} \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} -\frac{4}{3} & \frac{3}{4} & \frac{3}{4} \\ 0 & \frac{5}{4} & -\frac{5}{4} \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

where the column vectors of S are the eigenvectors, and the diagonal matrix D has the diagonals as the eigenvalues where the eigenvalue in each column matches with the eigenvector in the same column in S.
However, the S matrix is not orthogonal. But since we can just multiply each column vector in S by some constant and $S^{-1}AS = D$ still holds, we can just divide each column vector with their respective magnitude.

$$\begin{pmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\frac{4}{25} \\ 0 \\ \frac{3}{25} \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{4} \\ \frac{5}{4} \\ 1 \end{pmatrix} \to \begin{pmatrix} \frac{3}{50} \\ \frac{5}{50} \\ \frac{4}{50} \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ 1 \end{pmatrix} \to \begin{pmatrix} \frac{3}{50} \\ -\frac{5}{50} \\ \frac{4}{50} \end{pmatrix}$$

We get the new S as

$$S = \begin{pmatrix} -\frac{4}{25} & \frac{3}{50} & \frac{3}{50} \\ 0 & \frac{5}{50} & -\frac{5}{50} \\ \frac{3}{25} & \frac{4}{50} & \frac{4}{50} \end{pmatrix}$$

Since any orthogonal matrix inverse is equal to its transpose, we have

$$S^{1}AS \Leftrightarrow S^{T}AS$$

General Solution

1. Computing the determinant is fairly straightforward for any A. We just need to know how to use the Laplacian expansion formula.

2. To understand why solving the roots of the characteristic polynomial gives us the eigenvalues and kernel of the eigenspace. Let's derive the formulas from scratch to make this make more sense.
By definition of eigenvalues

$$Av = \lambda v$$

We can move the right to the left and get

$$Av - \lambda v = 0$$

We can factor v out and get

$$(A - \lambda I)v = 0$$

For a given λ, all the v that satisfies the first equation condition are just kernels of this matrix $A - \lambda I$. That is why the Eigenspace, the subspace of vectors that works for a given eigenvalue, is defined to be the kernel of some matrix.
We see that in order for such Eigenspace to have a dimension of non-zero, the matrix $A - \lambda I$ must be not invertible. One way to check if a matrix is not invertible is to see if the determinant is 0. So we try to solve for all the eigenvalues λ where the determinant is 0 so eigenspace of dimension non-zero can exist. The easiest way to systematically find the eigenvalues is to use the Laplacian expansion, and from such expansion, we get the characteristic polynomial.

3. The comprehensive way to check if a matrix is diagonalizable and/or invertible is listed in the first section of this guide.

4. To get an orthogonal matrix from any S in the $S^{-1}AS$ can be done by dividing each column by their respective magnitude.

Exercise 2. Final 2022.2.2

$$U = \{A \in \mathbb{R}^{2\times 2} | A \text{ has eigenvalue } 0\}$$

U is a subspace of $\mathbb{R}^{2\times 2}$ True/False?

Particular Solution

The surefire way to approach this kind of problem is to just check the three conditions that make a subset a subspace.

1. **Condition 1**: Check if the zero element exists in this subset.
   We see that
   $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
   exists

2. **Condition 2**: Check if the subset is closed under addition.
   For any $A, B \in U$, $A + B$ must also be in U. However, we can see that let A and B equal to
   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
   $$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
   We get
   $$A + B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
   which does not have an eigenvalue of 0, nor belong to the subset U.

3. **Condition 3**: Check if the subset is closed under scalar multiplication.
   For any $A \in U$, $\lambda \in \mathbb{R}$, λA must also be in U. We see that if a matrix has an eigenvalue of 0, then it must be not invertible. Any matrix that is not invertible multiplied by some λ will always be not invertible because the column vectors stay linearly dependent.

We see that a subset can only be a subspace if it satisfies all 3 conditions. This example only satisfies the first condition, therefore this subset is not a subspace.

General Solution

There are plenty of ways to check if a subset is a subspace. I would mostly recommend using the traditional 3-condition method as this is the most reliable, least risky, and can be applied to any variation of this category. There are also other methods for more specific types of problems in this category, such as converting the subset conditions into the kernel or image of a linear map which makes the proof a bit quicker. But if you are not too confident in your ability to identify the correct domain and codomain of such a linear map, the traditional way is the way to go.

Exercise 3. Final 2021.2.6

Let T be a differential operator of order n, $g \in P_n$. Then such $f \in P_n$ always exists for $T(f) = g$.

Particular Solution

We first try to understand what it means to be a differential operator of order n. This means that the function that is passed to the differential operator will have its n-th order derivative taken somewhere along in this functional. This means, that for any polynomial f, g of degree n, the polynomial will be reduced to a constant.
So if we just have $n = 2$ where

$$T = D^2$$
$$g = x$$

No matter what $f \in P_2$ that we pick, the result will always be a constant. Which can never equal g which has a degree of non-zero.
General Solution

We see that, as long as

$$\deg(f) - k \geq \deg(g)$$

then f will exist, where k is the smallest exponent in the characteristic equation.
This makes sense because this condition ensures the degree of f will equal to g after passing it through the differential operator. And two polynomial functions can only equal each other if they first meet the condition of equal degrees.

Exercise 4. Final 2023.3.2

Give an example of a differential operator T of order 4 with

$$\ker(T) = \operatorname{span}\{e^{-t}\cos(3t), e^{-t}\sin(3t), e^{3t}, te^{3t}\}$$

Particular Solution

In order to solve this, we first need to know the Theorem 20.10 in our lecture notes.

> **Theorem 20.10** *Let T be a differential operator. If $p_T(x)$ contains a factor $((x-a)^2+b^2)^m$, then*
>
> $$\{e^{at}\cos(bt), e^{at}\sin(bt), te^{at}\cos(bt), te^{at}\sin(bt), \ldots, t^{m-1}e^{at}\cos(bt), t^{m-1}e^{at}\sin(bt)\}$$
>
> *are $2m$ linearly independent elements in* $\ker(T)$.

We see that the elements that contain trigonometry functions are $e^{-t}\cos(3t), e^{-t}\sin(3t)$. By reverse engineering the theorem above, we can get the factor in the characteristic polynomial that contains those two as

$$((x-3)^2+3^2)^1$$

With the first two elements in the basis vectors dealt with, let's look at the second two. We see that the basis vectors are usually in the form of e^{at}, and will only be in the form of $t^k e^{at}$ if an eigenvalue has a multiplicity of $k+1$.
We see that the eigenvalue is 3 in this case and multiplicity of 2, so the factor that contains those two as

$$(x-3)^2$$

Together we get

$$\begin{aligned} p_T(x) &= ((x-3)^2 + 3^2)^1 (x-3)^2 \\ T &= ((D-x)^2 + 9)(D-3)^2 \end{aligned}$$

General Solution

This kind of problem is really about reverse engineering. We just need to pay attention to what forms each basis vector is in. There can be six different forms as we learned so far. Last four arise only when a factor of such form $((x-a)^2 + b^2)^m$ is in the characteristic polynomial.

$$e^{\lambda t}, t^k e^{\lambda t}, e^{at}\cos(bt), e^{at}\sin(bt), t^k e^{at}\cos(bt), t^k e^{at}\sin(bt)$$

Exercise 5. Final 2021.3.1

Give an example of an orthogonal matrix $B \in \mathbb{R}^{2\times 2}$ that is not diagonalizable.

Particular Solution

For me, it is easier to generate not diagonalizable matrices first, and then pick out the ones that are orthogonal. One such matrix that is not diagonalizable is the classic 2D rotational matrix.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

For simplicity, we can just set $\theta = \frac{\pi}{2}$. We get

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We see that such a matrix is not diagonalizable. Fortunately for us, our first example already satisfies the condition for being an orthogonal matrix. So we can finish here using this matrix as our example.

General Solution

To questions like giving an example of a matrix (not) invertible, (not) diagonalizable, (not) orthogonal, etc. It is often useful to refer to the four matrices in the first section of this guide. See if those matrices are enough to satisfy the conditions of a given problem. If they are not enough, we can use other well-known matrices with unique properties that might fit the conditions that are in another table in the same first section.

Exercise 6. Final 2022.3.3

Give an example of a matrix $H : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ which has eigenvalues $1, 2, 3, 4$.

Particular Solution

Before first considering such a linear map that exists in the set of possible H, it is often easier to first devise a matrix that satisfies the given conditions, which are the 4 unique eigenvalues in this case.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

We see that this diagonal matrix has four eigenvalues $1, 2, 3, 4$ correspondingly. Now we just need to reverse engineer to get a linear map that's from and to the vector space $\mathbb{R}^{2\times 2}$.
To do so, we just need to find a function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ that can transform our input from an arbitrary vector space to $\mathbb{R}^k$ and transform the output from $\mathbb{R}^k$ back to that same vector space. In other words, a function that makes the two vector spaces isomorphic.

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

We see that this function is a good and simple candidate to help us convert elements from and to $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$.
Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ and we have

$$\begin{aligned} Af(x) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\ &= \begin{pmatrix} a \\ 2b \\ 3c \\ 4d \end{pmatrix} \end{aligned}$$

$$f^{-1}(\begin{pmatrix} a \\ 2b \\ 3c \\ 4d \end{pmatrix}) = \begin{pmatrix} a & 2b \\ 3c & 4d \end{pmatrix}$$

We have found our linear map H, and it is

$$H(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & 2b \\ 3c & 4d \end{pmatrix}$$

General Solution

For this kind of question in general, like asking to find a linear map from and to some vector spaces while satisfying some conditions like determinant or eigenvalues equal to some numbers, dimension of kernel or image equal to some numbers, or diagonalizable, invertible, orthogonal, etc. It is easiest to first come up with an example in terms of a matrix (using the list of special matrices in section 1 of this guide as an aid), and then through a change of basis to get back to our original linear map.

Exercise 7. Final 2022.4

Consider linear map $F : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R}),\ F(f) = f''$.

1. Show that any $\lambda \in \mathbb{R}$ is an eigenvalue of F.

2. For each eigenvalue λ, calculate their corresponding eigenvectors

Particular Solution

1. It might be first daunting to look at this problem and not sure where to start. It is often useful to just plug in some numbers, and play around with the conditions given, to have a better grasp of what the questions are asking.
   For example, we can ask the question of can $\lambda = 1$ exists. In order for an eigenvalue to exist, it must have at least 1 eigenvector. Therefore, if we can find a function f such that $f = f''$, then $\lambda = 1$ can exist. One such function f is

$$f(x) = e^x$$
$$f''(x) = e^x$$
$$f(x) = 1 \cdot f''(x)$$

   Maybe just one number plugged in might not be enough, we can ask can $\lambda = 2$ exists. We see that if

$$f(x) = e^{\sqrt{2}x}$$
$$f''(x) = 2e^{\sqrt{2}x}$$
$$f(x) = 2 \cdot f''(x)$$

   Ok, from here we might form the conjecture that $\forall \lambda \in \mathbb{R}$

$$f(x) = e^{\sqrt{\lambda}x}$$

   this is each λ's corresponding eigenvector.
   However, when $\lambda < 0$, we see that this is no longer the case. So we can think of some other functions where $f = -\lambda \cdot f''$. One such example is this

$$f(x) = \sin(\sqrt{\lambda}x)$$

   Now we just showed that for any $\lambda \geq 0$ and $\lambda < 0$, we can find a corresponding eigenvector. Therefore, all eigenvalues $\in \mathbb{R}$ can exist.

2. Now the question is asking us to compute all the eigenvectors for each eigenvalue. So it is no longer as simple as guessing functions and seeing if they satisfy the condition $f = f''$ anymore. In order to compute all the possible eigenvectors, we need to approach this more systematically.
   Let's rewrite the condition more mathematically

$$f''(x) = \lambda f(x)$$

   Rearrange and we get

$$f''(x) - \lambda f(x) = 0$$

   Fortunately, we see that we have reduced our problem to something we can definitely solve. In this case, solving for the possible f in a linear differential equation.

$$x^2 - \lambda = 0$$

   We can write out its characteristic polynomial, solve for the roots, and get

$$x = \pm\lambda$$
$$f(x) = e^{\pm\sqrt{\lambda}x}$$

Since the degree of the characteristic polynomial is 2, it is expected that the dimension of our kernel is 2. Indeed, we see that the kernel is spanned by 2 different basis vectors as above. So we see that for each $\lambda \in \mathbb{R}$

$$E_\lambda(x) = \text{span}(e^{-\sqrt{\lambda}x}, e^{\sqrt{\lambda}x})$$

Though one edge case is that when $\lambda = 0$, we have the characteristic polynomial as

$$(x-0)^2 = 0$$

so instead of $e^{\pm 0 \cdot x}$, we have $e^{0 \cdot x}, xe^{0 \cdot x}$, or in other words

$$E_{\lambda=0}(x) = \text{span}(1, x)$$

We also observe that when $\lambda < 0$, $\sqrt{\lambda}$ becomes imaginary. We can use Euler's equation to expand such a function as a sum of sin and cos. But this step is not necessary.

General Solution

Every time when we face problems that involve differential operators or differential equations and ask us to solve for the possible $f \in C^\infty(\mathbb{R}, \mathbb{R})$. Our goal is just try to first reduce the problem into something we already know, in this case, reducing the problem to a linear differential equation, and use characteristic polynomials to solve for the roots and obtain the proper basis vectors of f for the subspace of all possible f satisfying such differential equation condition.

3 Closing

Congrats on making it to the end of this guide. Hopefully, you got something out of it. This is also the end of the last Linear Algebra course that the Nagoya G30 program has to offer. Making it to the end of the course is definitely not easy, but at least now you know what to reply to some random Sakae man on the streets right?

If questions/errors, contact me on Discord/Discord/Discord/Discord/Discord/Discord/Discord, in order of high $\rightarrow$ low priority. My profile picture is a pixel art of a Blue Bird for all of them.