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MP1 Midterm Cram

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> One can have an entire lunch, dinner, full-length sleep, watch the sun rise and fall, 5 stages of grief, existential crisis, ♪ it's just a dream ♪, and still be in the exam.
>
> - your dearest 10-hour weekend

Guide Content:

1. Classification of ODEs
2. Definitions & Properties
3. Theorems
4. Recipes

As usual, if you see any errors contact me. You know, I just realized for the past half of this semester, I didn't have a sleep schedule, I was in a sleep freestyle.
This cram list is definitely not exhaustive, though I tried covered the most of the definitions, theorems, and problem types that I think are important.
I recommend in addition to reading the theory from this guide, it is also more appropriate to glance through our tutorial question sheets. Not necessarily to solve every single question but at least enough to experience all the problem types and get a rough idea on how to solve each of them.

1 Classification of ODEs

Before we understand what theorems apply under what circumstances, or what recipes are used for which ODEs, it is probably better to first grasp the differences between all the ODEs we have learned so far. Below I listed the ODEs we learned in mostly chronological order.
For the following ODEs, we assume there exist intervals I_x, I_y for variables x, y where all the functions defined for a given ODE are continuous under them and the ODEs holds true $\forall x \in I_x, \forall y \in I_y$.

1. **1st + Linear**
$$p(x)y' + q(x)y = g(x)$$
2. **1st + Separable**
$$M(x) + N(y)y' = 0$$
3. **1st + Exact**
$$M(x,y) + N(x,y)y' = 0$$
$$\partial_y M(x,y) - \partial_x N(x,y) = 0$$
4. **1st + Inexact →1st + Exact**
$$M(x,y) + N(x,y)y' = 0$$
$$\partial_y \left(\mu(x,y)M(x,y)\right) - \partial_x \left(\mu(x,y)N(x,y)\right) = 0$$
5. **2nd + Linear + Homogeneous + Constant Coefficient**
$$ay'' + by' + cy = 0$$

6. **2nd + Linear + Homogeneous**

$$p(x)y'' + q(x)y' + r(x)y = 0$$

7. **2nd + Linear + Non-Homogeneous**

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

$$\exists x_0 \in I_x \text{ s.t. } g(x_0) \neq 0$$

8. **2nd + Linear**

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

9. **2nd + Linear + Homogeneous →2nd + Exact**

$$p(x)y'' + q(x)y' + r(x)y = 0$$

$$\text{where } (p(x)y')' + (f(x)y)' = 0$$

10. **2nd + Linear + Homogeneous →2nd + Inexact →2nd + Exact w/ Adjoint Equation**

$$p(x)y'' + q(x)y' + r(x)y = 0$$

$$\text{where } (\mu(x)p(x)y')' + (f(x)y)' = 0$$

$$\text{s.t. } P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$$

11. **nth + Linear + Homogeneous + Constant Coefficient**

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

12. **nth + Linear + Homogeneous**

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$$

13. **nth + Linear + Non-Homogeneous**

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = g(x)$$

$$\exists x_0 \in I_x \text{ s.t. } g(x_0) \neq 0$$

14. **nth + Linear**

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = g(x)$$

2 Definitions & Properties

Here are some definitions and properties that might be useful to know. Some of the properties mentioned here are direct results of certain theorems that will be talked about in the next section. But I decided to include those as ”properties” prior as I felt like they intuitively make more sense when bundled with the definitions.

Many of the definitions here such as linear independence, or fundamental set of solutions, etc, are actually not limited to just two solutions for a 2nd order linear ODE. Keep in mind that they be generalized to n-th order as well. Stay flexible.

1) **Linear Independence**

Two functions $y_1(x), y_2(x)$ defined on a common interval I are **linearly dependent** iff $\exists c_1, c_2 \in \mathbb{R}$ such that $\forall x \in I$ outside $c_1 = c_2 = 0$, we have

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

Two functions $y_1(x), y_2(x)$ defined on a common interval I are **linearly independent** iff $y_1(x), y_2(x)$ are not linearly dependent.

> Remark: Just because y_1, y_2 are linearly dependent on I_1 and I_2 does not imply y_1, y_2 must also be linearly dependent on $I_1 \cup I_2$. (See textbook Page 153, Problem 28)

2) **Wronskian**

Two functions $y_1(x), y_2(x)$ defined on a common interval I are differentiable has their **Wronskian** defined as

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

Functions $y_1(x), \cdots, y_n(x)$ defined on a common interval I are n-1 th differentiable has their **Wronskian** defined as

$$W(y_1, \cdots, y_n) = \det \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

We can also say $y_1(x), y_2(x)$ defined on a common interval I are differentiable are **linearly independent** if

$$\exists x_0 \in I, \text{ s.t. } W(y_1, y_2)(x_0) \neq 0$$

> Remark: The converse of the above statement is not always true. From the contrapositive, we see that if y_1, y_2 are linearly dependent, then Wronskian $0, \forall x_0 \in I$. This means that if y_1, y_2 are linearly dependent and differentiable on I_1, I_2 individually, then $W(y_1, y_2) = 0, \forall x_0 \in I_1 \cup I_2$, but y_1, y_2 can still be linearly independent on $I_1 \cup I_2$.

3) **Linear Operator**

Given a nth order linear homogeneous ODE in the general form in the list of ODEs from the previous section.
We can have **Linear Operator** of the above ODE that takes a function $y(x)$ as input defined as

$$L \coloneqq p_0(x)\frac{\partial^n}{\partial x^n} + p_1(x)\frac{\partial^{n-1}}{\partial x^{n-1}} + \cdots + p_n(x)$$

Due to the ODE being linear and homogeneous, we can have the property **Superposition**

$$L(a_1 y_1 + a_2 y_2 + \cdots + a_n y_n) = 0$$

$\forall a_1, \cdots, a_n \in \mathbb{R}$, where $y_1, \cdots, y_n$ are solutions to the ODE.

4) **Linear Independence for Solutions of ODE**

Given a 2nd order linear homogeneous ODE in the general form in the list of ODEs from the previous section.
Let all functions $p(x), q(x), r(x)$ be continuous on I, and y_1, y_2 be solutions of the ODE. $y_1(x), y_2(x)$ are **linearly dependent** on I if and only if

$$\exists x_0 \in I \text{ s.t. } W(y_1, y_2)(x_0) = 0$$

Similarly $y_1(x), y_2(x)$ are **linearly independent** on I if and only if

$$\exists x_0 \in I \text{ s.t. } W(y_1, y_2)(x_0) \neq 0$$

Remark: The definitions of linear independence mentioned in this guide are all logically equivalent. The only difference is that if $y_1, y_2 \in I$ are solutions of a single ODE, and if y_1, y_2 linearly dependent on I_1, I_2, then y_1, y_2 must also be linearly dependent on $I_1 \cup I_2$. This restriction allows the converse statement, or single Wronskian non-zero implies linear independence true.
In other words, testing two functions' linear independence using Wronskian might be sometimes unreliable. But if they are both solutions under a single ODE, then Wronskian $\Leftrightarrow$linear independence becomes logically equivalent and incredibly convenient to work with.
Assuming every point in the interval is ordinary.

5) **Fundamental Set of Solutions**

Given a 2nd order linear homogeneous ODE in the general form in the list of ODEs from the previous section.
Let all functions $p(x), q(x), r(x)$ be continuous on I, and y_1, y_2 be solutions of the ODE. The following are logically equivalent

1) y_1, y_2 are a **fundamental set of solutions** on I.
2) y_1, y_2 are linearly independent on I.
3) $\exists x_0 \in I$ s.t. $W(y_1, y_2)(x_0) \neq 0$.
4) $W(y_1, y_2)(x_0) \neq 0$, $\forall x_0 \in I$.

The **general solution** of this 2nd order linear homogeneous ODE is then the linear combination of the solutions in the fundamental set of solutions.

6) **Uniform & Point-Wise Convergence**

Given a sequence $\{f_n(x)\}$, it is **point-wise convergent** if $\exists k \in \mathbb{R}, \forall \epsilon > 0$ and $N \in \mathbb{N}$ depends on ϵ and x, such that

$$|f_n(x) - k| < \epsilon, \forall n \geq N, \forall x \in \mathbb{R}$$

It is **uniformly convergent** if N depends only on ϵ for the above to hold.

Remark: As we can see from the definition, uniformly convergent implies point-wise convergent, though the converse is false.

7) **Radius of Convergence**

Given a power series $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ centered at x_0. Its **radius of convergence** ρ is defined as $f(x)$ converges absolutely $\forall x \in (x_0 - \rho, x_0 + \rho)$ and diverges outside of $(x_0 - \rho, x_0 + \rho)$.
Refer to complex analysis lecture notes on how to calculate ρ in more detail. As a reminder, we can calculate ρ in three ways

$$\frac{1}{R} = \lim_{n\to\infty} \left(\sup_{m \geq n} |a_m|^{\frac{1}{m}} \right)$$

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

8) **Analytic**

A function $f(x)$ is **analytic** at x_0 if it can be represented as a Taylor series centered at x_0 that equals to $f(x)$ itself around a non-zero radius open ball. Recall that Taylor expansion at x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

In our math lectures, the above is the full definition. But for our physics lectures, we don't need to be entirely mathematically rigorous. So often just showing $f(x)$ have a Taylor series expansion at x_0 is enough.

9) **Ordinary & Singular Point**

   Given a 2nd order linear homogeneous ODE in the general form in the list of ODEs from the previous section.
   x_0 is an **ordinary point** the following two limits both exist

$$\lim_{x \to x_0} \frac{q(x)}{p(x)} \text{ and } \lim_{x \to x_0} \frac{r(x)}{p(x)}$$

   If x_0 is **singular point** if it is not ordinary.

3 Theorems

All the important theorems at a glance

1) **Existence& Uniqueness Theorem for nth + Linear ODE**

   Given a nth order linear homogeneous ODE in the general form in the list of ODEs from the previous section.
   Let all functions $p_0, \cdots, p_n$ be continuous in I. There exists an unique solution $y(x)$ for this given ODE that satisfies

$$y^{(i)}(x_0) = a_i$$

   $\forall i \in [0, \cdots, n]$, and $x_0 \in I$, for some $a_0, \cdots, a_n \in \mathbb{R}$.

2) **Existence& Uniqueness Theorem for 1st + Nonlinear ODE**

   Let f and $\partial_y f$ continuous on a rectangle $(\alpha, \beta) \times (\alpha', \beta')$, containing point (x_0, y_0). Then there exists an δ such that there exists an unique solution $y(x)$ that satisfies

$$y' = f(x, y),\ y(x_0) = y_0$$

   $\forall x \in (x_0 - \delta, x_0 + \delta)$.

3) **Theorem 3.2.3, 3.2.4, 3.2.5**

   If two solutions y_1, y_2 of a 2nd order linear ODE are linearly independent through the Wronskian test on an interval I where the coefficients of the ODE are continuous and ordinary, then the linear combination of those two solutions $c_1 y_1(x) + c_2 y_2(x)$ is still a solution.
   And if we also have the linear combination satisfy $y(x_0) = a_0, y'(x_0) = a_1$, then the constants c_1, c_2 can be found.
   Additionally, y_1, y_2 can form a fundamental set of solutions for this given ODE.

   Remark: This can easily be generalized to nth order linear ODE. Check Theorem 4.1.2. I will omit it here in this guide for condensation.

4) **Abel's Theorem**

   Given $y'' + p(x)y' + q(x)y = 0$, where p, q continuous on I, and y_1, y_2 solutions of this ODE. Then

$$W(y_1, y_2)(x) = C \exp\left(-\int p(x)\, dx\right)$$

   where c is a constant that depends on y_1, y_2.

Corollary: As we can see, e^x will never equate to zero, so the only way for Wronskian to be zero is for C to be zero. And if Wronskian is zero for some x_0 it will always be zero for any $x \in I$, this is true for non-zero as well. Therefore, it becomes incredibly easy to tell linearly independent and dependent solutions apart with the Wronskian due to this above theorem. Remark: One thing to note is that this Abel's formula is derived based on the fact y'' have a coefficient of 1, or in other words, the entire interval I contains only ordinary points. If we test the solutions y_1, y_2 on singular points, the above corollary will not hold.

Proof. We have

$$y_1'' + p(x)y_1' + q(x)y = 0$$
$$y_2'' + p(x)y_2' + q(x)y = 0$$

Multiply first equation by $-y_2$, second by y_1, combine them we get

$$(y_1y_2'' - y_1''y_2) + p(x)(y_1y_2' - y_1'y_2) = 0$$

Observe that

$$W'(y_1, y_2) = y_1y_2'' - y_1''y_2$$

So we can rewrite the combined equation as

$$W' + p(t)W = 0$$

And we can obtain the Abel's formula from here □

5) **General solution of non-homogeneous linear ODE**

Given a nth order linear non-homogeneous ODE in the general form in the list of ODEs from the previous section.
Let all functions $p_0, \cdots, p_n, g$ be continuous in I. Then the general solution can be written as

$$y(x) = y_h(x) + y_p(x)$$

where y_h is the general solution to the homogeneous version of the given ODE, and y_p is a any particular function that satisfies the non-homogeneous ODE. Additionally, the general solution y_h can be a linear combination a fundamental set of solutions and rewrite y as

$$y(x) = (c_1y_1(x) + \cdots c_ny_n(x)) + y_p(x)$$

6) **Fuch's Theorem**

Given a general 2nd order homogeneous linear ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0, then the two linearly independent solutions y_1, y_2 and general solution y can be expressed as

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^n$$
$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences of constants and c_1, c_2 variables.

Remark: If P, Q, R are all represented as polynomials, then instead of checking if the ratio are analytic, we can simplify condition to checking if $\lim_{x\to x_0} \frac{Q}{P}$ and $\lim_{x\to x_0} \frac{R}{P}$ exist. Additionally, let $\rho, \rho_1, \rho_2, \rho_3, \rho_4$ be the radius of convergence for $y, y_1, y_2, \frac{Q}{P}, \frac{R}{P}$. We have that $\rho = \min\{\rho_1, \rho_2\} \geq \min\{\rho_3, \rho_4\}$

4 Recipes

Make sure to understand the necessary pre-requisites before throwing a nuke on those types of problems. Observe many of the theorems and problems the recipes solve have only a constant coefficient of 1 in front of the term with the highest order (e.g. $1 \cdot y'' + p(x)y' + q(x)y = g(x)$ instead $p(x)y'' + q(x)y' + r(x)y = g(x)$). This important distinction makes sure that all functions are continuous on an interval I such that all points in this interval are ordinary. Freeing us from dealing with edge cases that singular points might bring (See textbook Page 153, Problem 27).
Some recipes below can also be generalized from 2nd to nth order linear ODEs.

1) (a) **Problem:** 1st + linear

$$y' + p(x)y = g(x)$$

(b) **Recipe:**

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

$$y(x) = \frac{\int \mu(x)g(x)\,dx + C}{\mu(x)}$$

(c) *Proof.* Multiply our ODE by $\mu(x)$ we get

$$\mu y' + \mu p y = g$$

Let's also define μ such that $\mu g = (\mu y)' = \mu' y + \mu y'$. Then we see that

$$\mu' y = \mu p y$$
$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

Using our definition, we can continue with

$$\mu(x)y(x) = \int \mu(x)g(x)\,dx + C$$

Divide both sides by $\mu(x)$ we get our recipe. □

2) (a) **Problem:** 1st + separable

$$M(x) + N(y)y' = 0$$

where $y(x_0) = y_0$

(b) **Recipe:**

$$\int_{x_0}^{x} M(x)\,dx + \int_{y_0}^{y} N(x)\,dx = 0$$

(c) *Proof.* Let $H_1'(x) = M(x), H_2'(y) = N(y)$. We can rewrite the ODE as

$$H_1'(x) + H_2'(y)\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial}{\partial x} H_1(x) + \frac{\partial}{\partial x} H_2(y) = 0$$
$$H_1(x) + H_2(y) = C$$

Since $H_1(x) = \int_{x_0}^{x} M(x)\, dx$ and $H_2(x) = \int_{y_0}^{y} N(y)\, dy$ by definition. We can obtain our recipe above. ☐

3) (a) **Problem:** 1st + exact

$$M(x,y) + N(x,y)y' = 0$$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(b) **Recipe:**

$$\phi(x,y) = C$$

where

$$\frac{\partial \phi}{\partial x} = M,\ \frac{\partial \phi}{\partial y} = N$$

(c) *Proof.* Refer to textbook Page 91 for proof. ☐

4) (a) **Problem:** 1st + inexact

$$M(x,y) + N(x,y)y' = 0$$
$$\exists \mu(x,y) \text{ s.t. } \frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$$

(b) **Recipe:**

$$\phi(x,y) = C$$

where

$$\frac{\partial \phi}{\partial x} = \mu M,\ \frac{\partial \phi}{\partial y} = \mu N$$

(c) *Proof.* Similar to the 1st + exact proof. ☐

5) (a) **Problem:** 2nd + linear + homogeneous + constant coefficient

$$ay'' + by' + cy = 0$$

(b) **Recipe:**

Let r_1, r_2 be solutions of $ax^2 + bx + c = 0$.
If $r_1 \neq r_2 \in \mathbb{R}$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

If $r_1 = r_2 \in \mathbb{R}$

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

If $r_1, r_2 = \lambda \pm \mu x i \in \mathbb{C}$

$$y(x) = e^{\lambda x} \left(c_1 \cos(\mu x) + c_2 \sin(\mu x)\right)$$

(c) *Proof.* Assume solution is $y(x) = e^{rx}$. Plug this into the ODE we get

$$e^{rx}\left(ar^2 + br + c\right) = 0$$

Since e^{rx} cannot be equal to zero. Only $ar^2 + br + c = 0$ is possible. Solving for the roots of this quadratic we will get three different scenarios as follows in the recipe. ☐

6) (a) **Problem:** Reduction of Order from 2nd →1st

$$y'' + p(x)y' + q(x)y = 0$$

Suppose we know one solution $y_1(x)$ for ODE above, find second solution.

(b) **Recipe:**

We have $y_2(x) = v(x)y_1(x)$, with $v(x)$ a non-constant function to ensure y_1, y_2 linearly independent, where $v'(x) = w(x)$ such that

$$y_1(x)w' + (2y_1'(x) + p(x)y_1(x))w = 0$$

(c) *Proof.* We have that

$$\begin{aligned} y_2 &= vy_1 \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Plug those into the ODE we get that

$$(v''y_1 + 2v'y_1' + vy_1'') + p\,(v'y_1 + vy_1') + q(vy_1) = 0$$

Reordering this we get

$$(v''y_1 + 2v'y_1') + p\,(v'y_1 +) = -v\,(y_1'' + py_1' + qy_1)$$

Since the right hand side is just zero. Set $v' = w$, reorder the above ODE we get the condition in the recipe. We see that this new ODE in terms of W is a first order separable and linear ODE. We can then solve it by either means. □

7) (a) **Problem:** 2nd + linear + Variation of Parameters

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

Suppose we know $y_1(x), y_2(x)$ for homogeneous version of the ODE above, find particular solution for the above non-homogeneous ODE.

(b) **Recipe:**

$$y_p(x) = \mu_1(x)y_1(x) + \mu_2(x)y_2(x)$$

where

$$\mu_1(x) = -\int \frac{y_2(x)\frac{g(x)}{p(x)}}{W(y_1, y_2)(x)}\,dx + C_1$$

$$\mu_2(x) = \int \frac{y_1(x)\frac{g(x)}{p(x)}}{W(y_1, y_2)(x)}\,dx + C_1$$

(c) *Proof.* The beginning of this proof is similar to the one from Reduction of Order. We assume that $y_p = \mu_1 y_1 + \mu_2 y_2$ for some μ_1, μ_2. We see that

$$\begin{aligned} y_p' &= \mu_1' y_1 + \mu_1 y_1' + \mu_2' y_2 + \mu_2 y_2' \\ y_p'' &= (\mu_1' y_1 + \mu_2' y_2)' + \mu_1' y_1' + \mu_1 y_1'' + \mu_2' y_2' + \mu_2 y_2'' \end{aligned}$$

Plug those into the original ODE and reorder, we see that

$$p\,((\mu_1' y_1 + \mu_2' y_2)' + \mu_1' y_1' + \mu_2' y_2') + q\,(\mu_1' y_1 + \mu_2' y_2) = g - (\mu_1(py_1'' + qy_1' + ry_1) + \mu_2(py_2'' + qy_2' + ry_2))$$

We notice that the right hand side is just g as the rest are just zero since y_1, y_2 are solution to the homogeneous version of this ODE.

$$p\left((\mu_1' y_1 + \mu_2' y_2)' + \mu_1' y_1' + \mu_2' y_2'\right) + q\left(\mu_1' y_1 + \mu_2' y_2\right) = g$$

Since μ_1, μ_2 are arbitrary, we can manually set them to whatever we like so long the above equality is satisfied. One such solution set is the one below

$$\mu_1' y_1 + \mu_2' y_2 = 0$$
$$\mu_1' y_1' + \mu_2' y_2 = \frac{g}{p}$$

We can rewrite those as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{g}{p} \end{pmatrix}$$

Solving for μ_1', μ_2' we get that

$$\begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} -y_1 \frac{g}{p} \\ y_1 \frac{g}{p} \end{pmatrix}$$

Finally take this matrix multiplication apart and integrate both μ_1', μ_2' individually we get the recipe presented. □

8) (a) **Problem:** nth + linear + Undetermined Coefficients

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

Suppose we know $y_1(x), y_2(x)$ for homogeneous version of the ODE above, find particular solution for the above non-homogeneous ODE.

(b) **Recipe:**

TABLE 3.6.1 The Particular Solution of $ay'' + by' + cy = g_i(t)$

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t} \begin{cases} \sin \beta t \\ \cos \beta t \end{cases}$	$t^s[(A_0 t^n + A_1 t^{n-1} + \cdots + A_n)e^{\alpha t} \cos \beta t$ $+ (B_0 t^n + B_1 t^{n-1} + \cdots + B_n)e^{\alpha t} \sin \beta t]$

Notes. Here s is the smallest nonnegative integer ($s = 0, 1,$ or 2) that will ensure that no term in $Y_i(t)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, s is the number of times 0 is a root of the characteristic equation, α is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

Remark: In the case when our $Y(t)$ is part of the Kernel of the ODE's corresponding Linear Operator, we just keep multiplying everything by t until we are not.

(c) *Proof.* Proof should be fairly intuitive. Refer to textbook Page 176. □

9) (a) **Problem:** 2nd + linear + Series Solution w/ Fuch

$$p(x)y'' + q(x)y' + r(x)y = 0$$

Find general solution $y(x)$ of the above ODE at an ordinary point x_0. Additionally p, q, r are represented as polynomials.

(b) **Recipe:**

1) Check x_0 is an ordinary point.
2) Assume $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ and substitute into the ODE.
3) Obtain recurrence relation for $\{a_n\}$.
4) Find two linearly independent series solution y_1, y_2. Write the first four non-zero terms of each solution. Write in summation form if possible.
5) Find the radii of convergence ρ_1, ρ_2 for y_1, y_2. We have $y(x) = c_1 y_1(x) + c_2 y_2(x)$ radius of convergence $\rho = \min\{\rho_1, \rho_2\}$.

(c) We will show the converse, if y can be represented as a series solution, then the two ratios must be analytic.

Proof. Assume $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ centered at x_0. We see that

$$a_n = \frac{y^{(n)(x_0)}}{n!}$$

In other words, to uniquely identify y, we just need all of its infinite sequence of coefficients. And to find those coefficients we just need its infinite orders of derivatives. Is it possible to figure out those derivatives from our ODE? Turns out we can if

$$u_1(x) \coloneqq \frac{q(x)}{p(x)}$$
$$u_2(x) \coloneqq \frac{r(x)}{p(x)}$$

Since we are working on an interval I that $\forall x \in I$, x is ordinary. So we can rewrite ODE as

$$y'' = -\left(u_1 y' + u_2 y\right)$$

We see that if we know y, y', we can determine y''. Take derivative on both sides we get

$$y''' = -\left(u_1' y' + u_1 y'' + u_2' y + u_2 y'\right)$$

We see that if we know y', y'', we can determine y''', and so on. So basically as long as u_1, u_2 are k-th differentiable, then we can obtain the $k+1$-th derivative of y, and figure out a_{k+1}.
If we want to know the entirety of $\{a_n\}$, we need entirety of $y^{(n)}$, or the entirety of $u_1^{(n)}, u_2^{(n)}$. In other words, we want u_1, u_2 to be infinitely differentiable, and that basically requires them to be analytic. □

5 Closing

Congrats on making it to the end of this guide! Phew it sure ain't easy nowadays.
Hopefully, you got something out of it. If questions/errors, contact me on Discord/Instagram/Messenger/Wechat/Slack/Line/Telegram, in order of high → low priority. My profile picture is a pixel art of a Blue Bird for all of them.